The Stability and Safe Dimensioning of Circular Masonry Arches

A.L. Mrema, Lecturer, Department of Civil Engineering,
University of Dar es Salaam, Tanzania
P.O. Box 35131 Dar es Salaam, Tanzania

B.L.M. Mwamila, Associate Professor,
Department of Civil Engineering,
University of Dar es Salaam,

ABSTRACT: An analysis of masonry arches where collapse is assumed to occur by the hinging of voussoirs upon each other is made. The behaviour of the arch material is assumed to be such that the plastic theory is valid. Certain aspects related to the properties of the line of thrust are clarified. So far the approach by writers has been based on the conventional view that the line of pressure is tangential to the line of thrust. It is shown that this is not the case in general. Definitions are given on what is meant by "line of thrust" and line of pressure in arches and the analysis is done in the light of the meanings attached to these terms. The analysis is done for the case of a semi-circular masonry arch under an infinite overburden pressure and the results obtained are compared to ones obtained by Irvine (4) whose approach was based on the conventional views of the line of thrust and pressure. It is shown that there is a significant difference in results with the two approaches. The analysis is repeated for the case of masonry arches of a general angle of embrace.

The results obtained give critical values of voussoir thicknesses for the stability and the safe dimensioning of masonry arches.

1.0 INTRODUCTION

The art of building arches dates back to the Romans who used arches extensively in the construction of their bridges and aqueducts. The arches so constructed consisted of semi-circular rings with individual shaped stones. The Romans were so skilled that some of their arches still stand today.

The Romans were, however, handicapped by the fact that there was no theory at their time of determining safe dimensions for the arches. The mechanical principles of their stability were unknown and the builders relied mainly on their experience.

The Roman methods were gradually forgotten in the medieval times when there was little or no bridge construction.

Towards the end of the seventeenth century some attempts were made by French engineers to draw up some theories which would govern arch designs. The most notable scientist was Lahire (1640-1718) who applied the principles of statics in the solution of arch problems. Much is attributed to his book "Traite de Mechanique" (5) in which he applied the principles of the funicular polygon for the first time in arch analysis. His approach was to divide the arch into wedges and then determine the weights of the wedges necessary to ensure the stability of the whole structure. Lahire was able to extend the theory to determine the proper dimensions of the pillars supporting the arch (5).

Belidor, Perronet and Chezy are said to have been among the first to put Lahire's method into practical use (6). They used it to prepare tables for use in the calculation of thicknesses of arches.

Lahire's approach assumed the interacting surfaces of the voussoirs (wedges) to be smooth and that the pressure acted normal to the face of each voussoir. This led to the conclusion that the voussoir must be of increasing depth towards the springings for the arch to be in equilibrium.

Experiments are reported (10) of two arch models with polished metal voussoirs in which the pressure normal to the face of certain voussoirs was measured in order to prove the validity of the wedge theory.

An alternative approach adopted was to assume that the voussoirs were infinitely rough so that the arch fails by the rotation of some of the voussoirs about their edges. This was actually shown to occur in model experiments (1) in 1730. It was erroneously concluded that the joints of rapture for all the arches occured at the keystone and at 30° to the horizontal. Couplet (1) computed minimum voussoir depth necessary for the line of thrust to just lie within the arch. What is notable is that he recognized that the line of thrust in an arch consisting solely of a semi-circular ring of voussoirs insn't itself a semi-circle. He appeared to have had a clear understanding of lines of thrust and the mechanics by which arches fail.

The notion of "a line of pressure" and "a line of resistance" were introduced by Gerstner (2) in 1830 and is said to have done a lot of investigations of lines of pressure. Moseley (7) in 1839 made an important contribution by showing for the first time the difference between pressure lines and resistance lines and showed that they are different curves. Though this important distinction was made reasonably long ago it appears that it isn't fully appreciated by some people even today.

In recent years Pippard and Baker (9) have examined in detail the collapse of the voussoir arch. Heyman (3) has worked on the stability of masonry arches using the plastic theory and the lower bound theorem. Irvine (4) has considered the conditions obtaining when a classical semi-circular masonry arch is on the verge of collapse.

A fuller account of the historical development of arch theories and some useful references are contained in a book by S.P. Timoshenko (11) and a research paper by A.J.S. Pippard and L. Chitty. (10).

2.0 THEORY OF MASONRY ARCHES

2.1 Assumptions

Masonry arches may be constructed of masonry or brickwork where small units, bricks or voussoirs are held together by cement mortar. In this respect the arch structure can be considered to be made of composite material.

From the theoretical point of view it is important to distinguish between the two. A homogeneous structure may reasonably be expected to behave elastically within the limits set by the material of which it is made whereas the same assumption becomes questionable in the case of a structure made of composite material.

Brickwork, masonry and plain concrete though very strong in compression are relatively weak in tension. The presence of joints between bricks and voussoirs made of such materials will lead to cracks along the joints in regions of tension. It is reasonable in this case to use plastic theory and limit state in the design.

The application of the plastic theory to masonry arches is valid under the folowing assumptions:

i) Sliding failure doesn't occur.

It is assumed that the material is assembled in such a way that friction and the interlocking of parts will prevent disintergration by sliding. The shearing resistance of joints is neglected. The function of any mortar across a joint would be to facilitate a uniform distribution of thrust across the joint and not adhesive resistance.

ii) The material has an infinitely high compressive strength. This comes from the fact that stress levels in real arches are small when compared to their compressive strength.

iii) The tensile strength is small enough to be ignored. This assumption is conservative. In masonry construction though the stone might have a significant tensile strength, the mortar between the voussoirs is very weak and cracking would occur at the joints.

iv) The material is incompressible.

2.2. The Stability and Failure Mechanism

The masonry arch consists of a number of wedge shaped bricks of masonry with a jointing material of mortar or cement. The names used in connection with the arch ring and the adjacent parts are shown in Fig. 1.

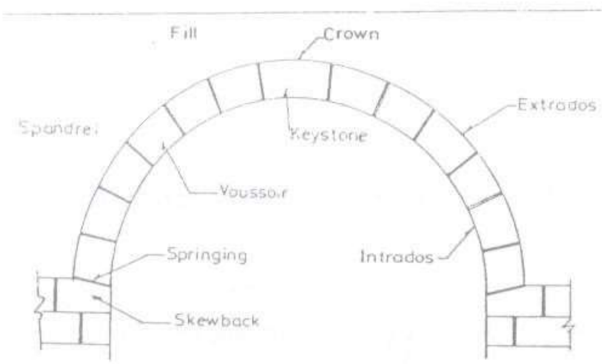

Fig. 1: Names Used in Masonry Arches

The wedge shaped bricks of which the arch is built are known as voussoirs. They are normally symmetrically disposed about a central voussoir called the keystone. The keystone is mainly for easthetic reasons and has no more structural importance than any other voussoir. The skewback is the block on the abutment upon which the end voussoirs rest. The highest point of the arch is the crown. The under surface of the structure is the soffit. When this form of arch is used for a bridge the spaces between the top of the arch and the level of a roadway is built up by a fill.

The analysis of masonry arches is a classical problem. By elastic considerations the structure is statically indeterminate. If the spandrel is filled with loose material the incidence of the load on the ring is indeterminate because the pressure of a granular material will not be wholly vertical but will have a horizontal component which is a function of the angle of repose. The line of pressure cannot therefore be easily determined (4).

The application of plastic principles is therefore of considerable interest.

Heyman (3) has considered the conditions under which the stability of the structure (Fig.2) is lost assuming the arch to be carrying only its own weight.

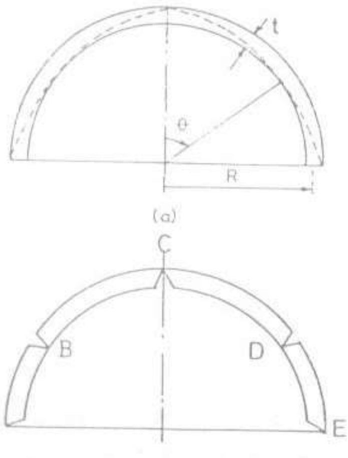

Fig.2: Line of thrust and mechanism of collapse.

He maintains that this occurs when the line of thrust (dotted in Fig.2) touches the extrados at the crown and the intrados at points on the haunches and passes through the extrados at the springings. The arch would then fail by the hinging of the voussoirs at points A through E.

The main problem is therefore to determine the thickness ratio t/R of the arch under this limiting condition and the angle θ at which the hinge forms.

Irvine (4) also considered the stability of the Roman arch under an infinite and variable overburden along the same lines as Heyman. His assumption was, however, that the line of pressure is tangential to the line of thrust which is generally not the case.

2.3 Formulation of the Arch Thrust Line Problem.

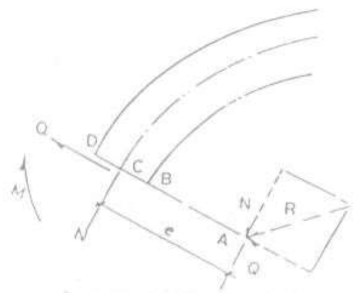

Fig. 3: Force actions at an arch section.

Consider an arch section as shown in Fig.3. The actions at any normal cross section BD can be resolved into a bending moment M, a shearing force Q and a normal force N through the centroid. The resultant of the forces N and Q is given by R,viz N + Q = R. The three forces (M,N,Q) are statically equivalent to a single force R acting at a distance (eccentricity) e, given by the ratio M/N.

The line of thrust is the locus of the eccentricity (e = M/N) with respect to the arch centre line, while the line of pressure is the envelope to R. The conventional view is that R is tangential to the line of thrust, (4). But R is not tangential to the line of thrust in general.

3.0 THE STABILITY OF THE SEMI-CIRCULAR MASONRY ARCH

3.1 Stability under Distributed Loading

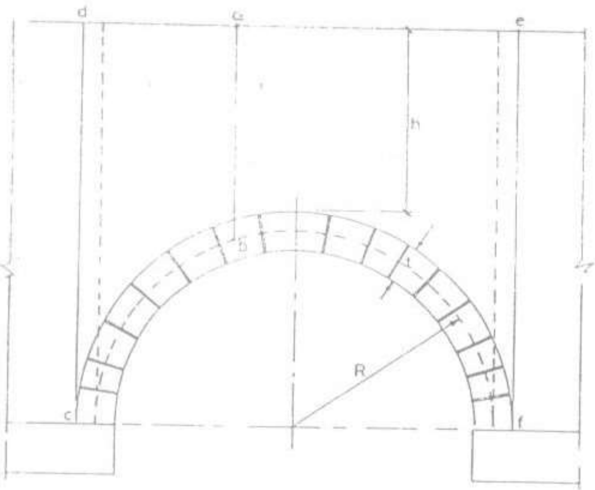

Fig.4: Stability under distributed loading.

Consider the case of an arch under an overburdern pressure. The magnitude of the distributed load will be proportional to the length ab and it will be assumed that the load acts over the centre-line of ring with mean radius R (Fig.4). Generally, the thickness t will be small in comparison to the radius R and the total load will not be very different from the actual load enclosed in cdef.

Assuming the plastic theory of failure and the mechanism shown in Fig.2 the problem will be to determine the critical thickness t and the angle θ when the arch is on the verge of collapse.

Consider an element of the arch as shown in Fig.5.

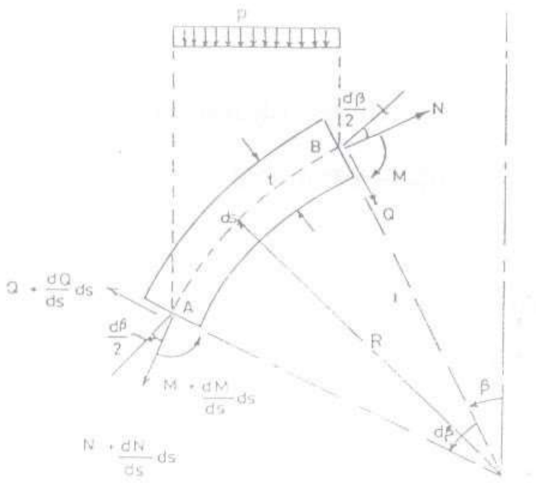

Fig. 5: Equilibrium of an arch element.

The equilibrium of the element requires the following:

i) In the tangential direction:

$$\frac{dN}{ds} + Q\,\frac{d\beta}{ds} + p\cos\beta\,\sin\beta = 0 \tag{1}$$

ii) In the radial direction

$$- \frac{dQ}{ds}\, ds + N d\beta + pds \cos^2\beta = 0 \qquad (2)$$

iii) Moments about A

$$\frac{dM}{ds} - Q - pds\, \frac{\cos^2\beta}{2} = 0 \qquad (3)$$

The solution of the above equations can be shown (8) to be

$$N = A \sin\beta + B \cos\beta - pR \sin^2\beta \qquad (4)$$

$$Q = - A \cos\beta + B \sin\beta + pR \sin\beta \cos\beta \qquad (5)$$

$$M = - AR \sin\beta - BR \cos\beta - p\frac{R^2}{4} cos 2\beta + C \qquad (6)$$

where A,B,C are arbitrary constants.

The boundary conditions on the verge of collapse (Fig.2) require

i) $At\ \beta = 0, \quad e = t/2.$

ii) At $\beta = \theta$ *(where the hinge forms at the intrados), de/ds* = 0.

iii) At $\beta = \pi/2$, $e = t/2$.

iv) At $\beta = \theta$, $e = -t/2$.

From symmetry of loading, the shear force at the crown should be zero.

From the above boundary conditions we obtain

$$\frac{t}{R} = \frac{\cos^2\theta}{2\cos\theta - 1} - 2 \qquad (7)$$

$\theta = 54.695^\circ$

$T = t/R = 0.1431$

The abutment thrust, H_h

$H_h = -0.5335pR$

The shear force at the intrados hinge position is given by

$Q = 3.64 \times 10^{-2} pR$

The normal force, N at the hinge position is

$N = -0.974\ pR$

The eccentricity is given by $e = M/N$ so that

$$\frac{de}{ds} = \frac{Q(1+eK) + ep\cos\beta\,\sin\beta}{N} \tag{8}$$

where K = 1/R is the curvature.

Under distributed loading p, we have at a hinge section.

$$Q = \frac{-ep\cos\beta\,\sin\beta}{(1+eK)} \tag{9}$$

The line of pressure is tangential to the line of thrust when Q=0. Clearly Q in eqn. (9) can never be zero under distributed loading p as conventionally assumed . The line of pressure cannot therefore be tangential to the line of thrust . The formation of a hinge doesn't depend only on the moment as the case is in beams but on both the moment and the normal force at the section.

It so happens that in the case of an infinite overburden pressure the shear force is small in comparison to the normal force and that there is little error in assuming the shear force is zero. For example Irvine (4) in assuming the shear force is zero came out with values of

θ = 54.7
t/R = 0.144
H_h = 0.536 pR

Irvine's values for small overburden heights significantly deviates from values obtained under the current approach (8).

3.2 Stability under the General Case of Overburden Height

Under the general case of overburden height h/R (Fig.6) values of t/R are obtained as shown in Fig.8. These values are compared with those obtained by Irvine whose approach was based on the conventional method.

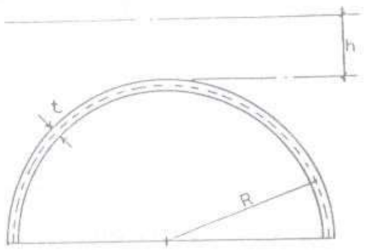

Fig. 6: Stability under the general case of overburden height.

3.3 Stability under the Case of an Infinite Overburden and a General Angle of Embrace.

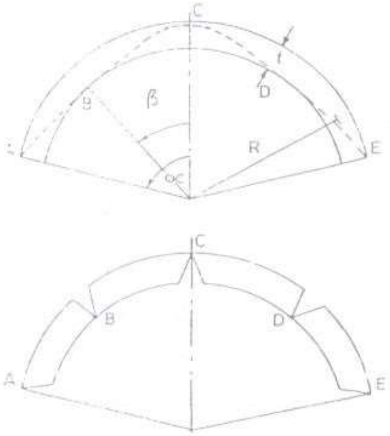

Fig.7: Failure under the general angle of embrace α and an infinite overburden.

Under a general angle of embrace α (Fig.7) and an infinite overburden analysis by the previous approach gives the values shown in Table 1. The critical voussoir thickness t/R, the horizontal component of pressure H_h/pR and the location of the hinge θ are given as a function of α.

These values are plotted in figures 9,10 and 11.

Table 1: Values of the critical voussoir depth t/R, the location of the hinge θ and the horizontal component of pressure H_h as a function of the angle of embrace.

α(deg.)	t/R	θ (deg.)	H_h/pR
20	0.0005	14.00	0.97
25	0.0011	17.41	
30	0.0022	20.75	0.95
35	0.0041	25.03	
40	0.0069	27.24	0.89
45	0.0108	30.36	
50	0.0162	33.40	0.83
55	0.0233	36.36	
60	0.0323	39.23	0.76
65	0.0435	42.02	
70	0.0572	44.71	0.69
75	0.0738	47.33	
80	0.0934	49.87	0.61
85	0.1164	52.32	
90	0.1431	54.69	0.53

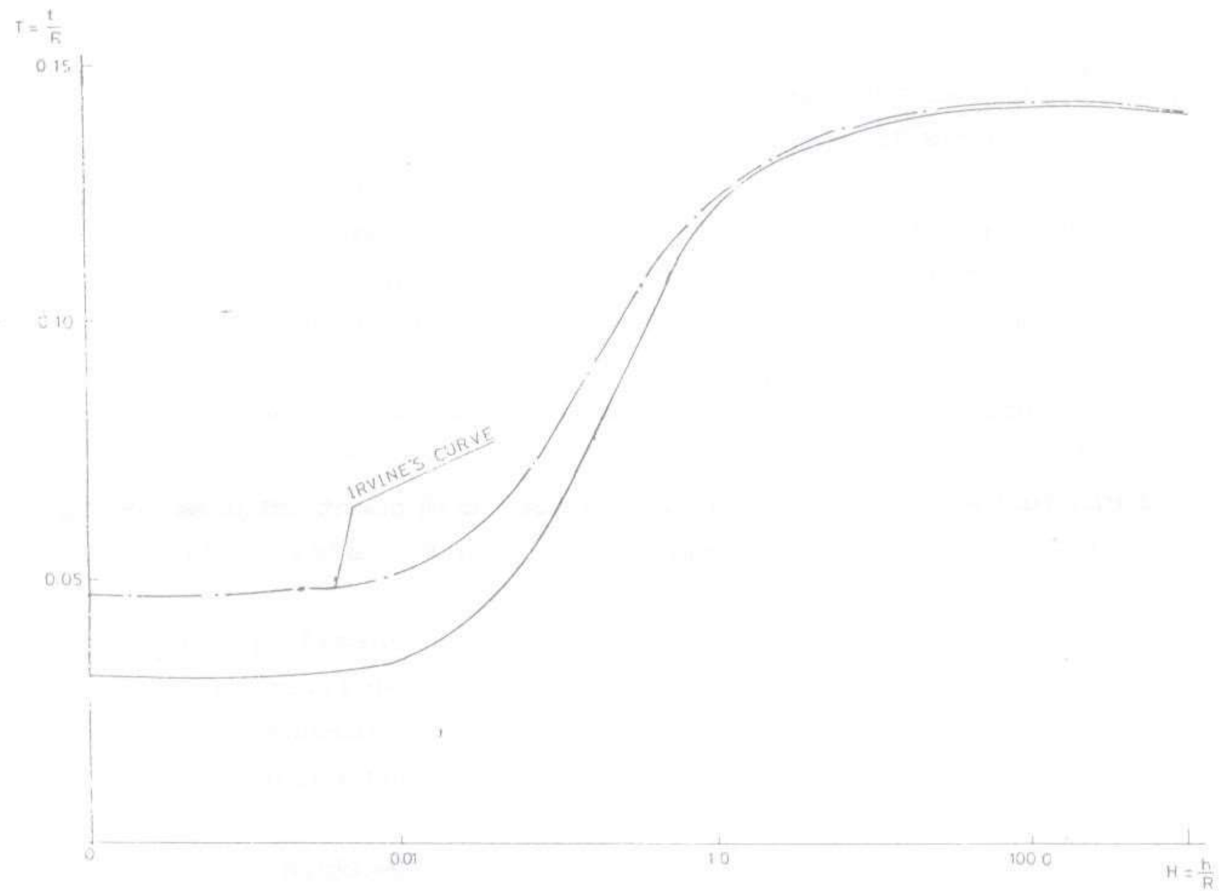

Fig. 8: Critical voussoir thickness T = t/R as a function of the overburden height, H = h/R (α = 90 deg.)

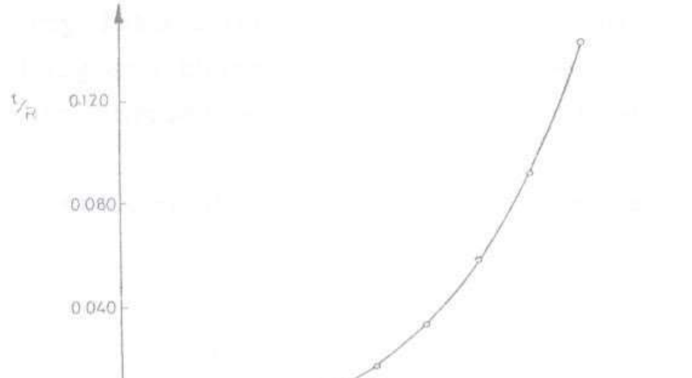

Fig. 9: Variation of voussoir thickness T = t/R with arch angle of embrace α. (h/R = ∞)

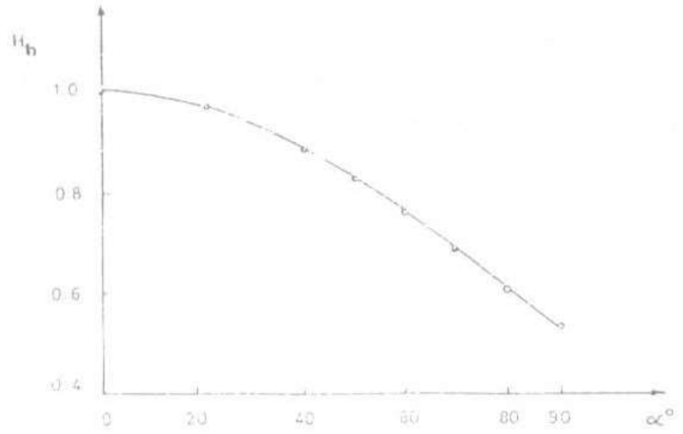

Fig. 10: Variation of the horizontal component of pressure H_h with arch angle of embrace α. ($h/R = \infty$)

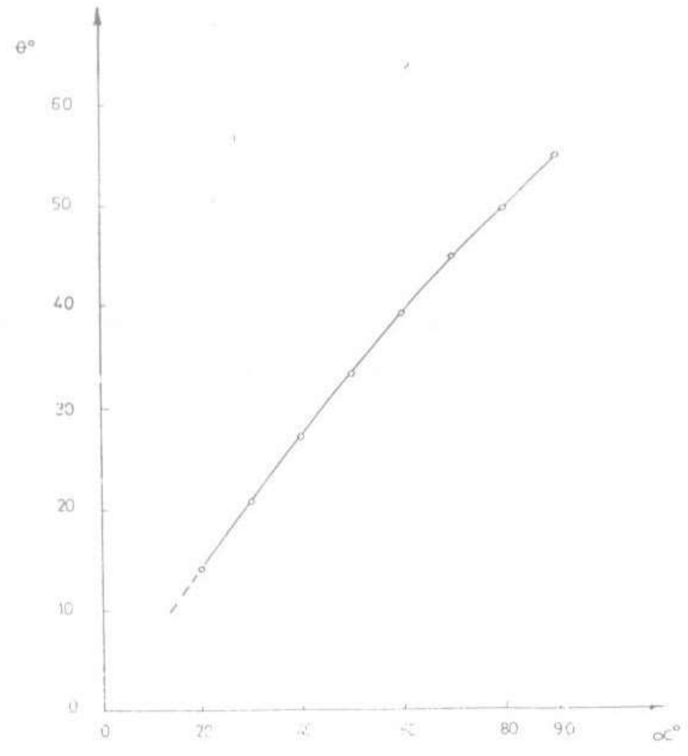

Fig. 11: Variation of the hinge angle θ with arch angle of embrace, α. ($h/R = \infty$)

4. CONCLUSIONS

An attempt has been made to clarify some aspects related to the properties of the line of thrust in the analysis of masonry arches. While the line of thrust is the locus of the eccentricity (e = M/N), the line of pressure is the envelope to R (Fig.3). The line of pressure is not generally tangential to the line of thrust except in sections where the shear force is zero.

Most of the previous work on the analysis of masonry arches assume failure to occur at sections of maximum moment just as it does in flexural members. It has been shown here that using the plastic theory failure will occur at sections of maximum eccentricity.

There is a significant deviation in results with the two approaches especially at small overburden pressures. Irvine(4) over-estimated the critical voussoir thickness by up to 30% and under estimated the angle locating the hinge position by about 10%.

The voussoir thickness calculated are minimum thicknesses if the arch has to stand. However, accidental super imposed loads will distort the line of thrust so that it can no longer be contained within the masonry. Arches must therefore be made thicker to take this possibility into account. In practice arches are made much thicker than they are theoretically required. The added advantage is that abutment thrusts are greatly reduced (4).

The results in this study can be used in the safe dimensioning of masonry arches.

REFERENCES

1. De Coulomb, C.A.; (1776), Memoires de mathematique et de physique, presentes a l'Academic Royale des Sciences, pars divers savans et lus dans ses assembless, 1773, p 343, Paris.

2. Gerstener, A: (1831), Handbuch der Mechanik, Vol.1, Prague.

3. Heyman, J, (1969), The ¯Safety of Masonry Arches, International Journal of Mechanical Sciences, 11, p 363-85.

4. Irvine, H.M.; (1979), The Stability of the Roman Arch, International Journal of Mechanical Sciences, 21, p 471-74.

5. De Lahire, P; (1731), Memoires de I'Academie Royale des Sciences, 1712, p 69, Paris.

6. Lasage, M., (1810), Recuil des Memoires extraits de la bibliotheque des Ponts et Chaussees, Paris.

7. Moseley, H.; (1839), A Treatise of Mechanics Applied to the Arts.

8. Mrema, A.L.; (1981), Arch Analysis, M.Sc. Dissertation, University of Strathclyde, Glasgow.

9. Pippard, A.J.S.; and Baker, J.F.; (1957), Engineering Structures, 3rd Edition, Chapter 16, Edward Arnold Publishers, London.

10. Pippard, A.J.S. and Chitty, L.; (1951), A study of the Voussoir Arch, National Building Studies Research Paper No. 11, D.S.I.R. H.M. Stationery Office.

11. Timoshenko, S.P.; (1952), History of Strength of Materials p. 62-66, p.211-15 McGraw Hill Publication, London.