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A DERIVATION OF A NOVEL EXPRESSION FOR THE INSTANTANEOUS EARTH-SUN DISTANCEUSING KEPLER'S LAWS AND SOME GEOMETRICAL RELATIONSHIPS

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ABSTRACT

In almost all solar energy calculations, knowledge is required about the earth-sun distance d and the mean earth-sun distance d_0 which is the Astronomical Unit (AU). Up to now most expressions of the the earth-sun distance have been approximate ones. For example, Spencer[1] gives $(d_0/d)^2$ as a trigonometric series function of the day angle Γ obtained by performing a Fourier series analysis of the ratio $(d_0/d)^2$ as Γ is varied. Furthemore, Duffie and Beckman [2] and Iqbal [3] all agree that for most engineering and technological applications an expression of $(d_0/d)^2 = 1+0.033Cos(2\,n/365)$ is adequate. The expr ession by Spencer depends on a numerical derivation whose improvement is difficult. The expression by Duffie et al. on the other hand appears to be obtained by fitting some data which again its improvement would depend a lot on the type of data used. In this paper therefore it is shown that a novel expression can be derived using only the Kepler's laws and plane geometrical considerations of the earth's orbit around the sun.

INTRODUCTION

In almost all solar energy calculations, the instantaneous earth-sun distance d and the Astronomical Unit (AU) d_0 (the mean earth-sun disatance) is used. Up to now the expressions of the earth-sun distance d have been approximate ones. For example, Spencer [1] gives $(d_0/d)^2$ as a trigonometric series function of the day angle Γ obtained by performing a Fourier series analysis of the ratio $(d_0/d)^2$ as Γ is varied. Another expression given by Duffie and Beckman [2] and Iqbal [3] is said to be adequate for most engineering and technological applications like those in solar thermal heating processes or photovoltaic design processes and is given by $(d_0/d)^2$

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$= 1+0.033\mathrm{Cos}(2\pi n/365)$. Where as the first expression by Spencer depends on a numerical derivation and is difficult to improve, the second one by Duffie and Beckman [2] and Iqbal [3] is also obtained by fitting the measured data and thus principally not different from the first one. Though one would expect that the two values obtained using different approaches with the same data would give the same values, surprisingly the two differ. In this paper a novel expression is derived using only Kepler's laws and plane geometrical considerations of the earth's orbit around the sun in an effort to solve the above discrepancy with the assumption that Kepler's laws apply where the earth's orbit is elliptical and furthermore taking the period with which the earth revolves around the sun as 365.25 days.

PRESENT EXPRESSIONS FOR THE EARTH-SUN DISTANCE D

The earth revolves around the sun in an elliptical orbit with the sun at one of the foci in accordance with Kepler's First Law (see Figure 1). The mean sun-earth distance d_o has been given the name of One Astronomical Unit [1 AU] where 1 AU = 1.496×10^8 km.

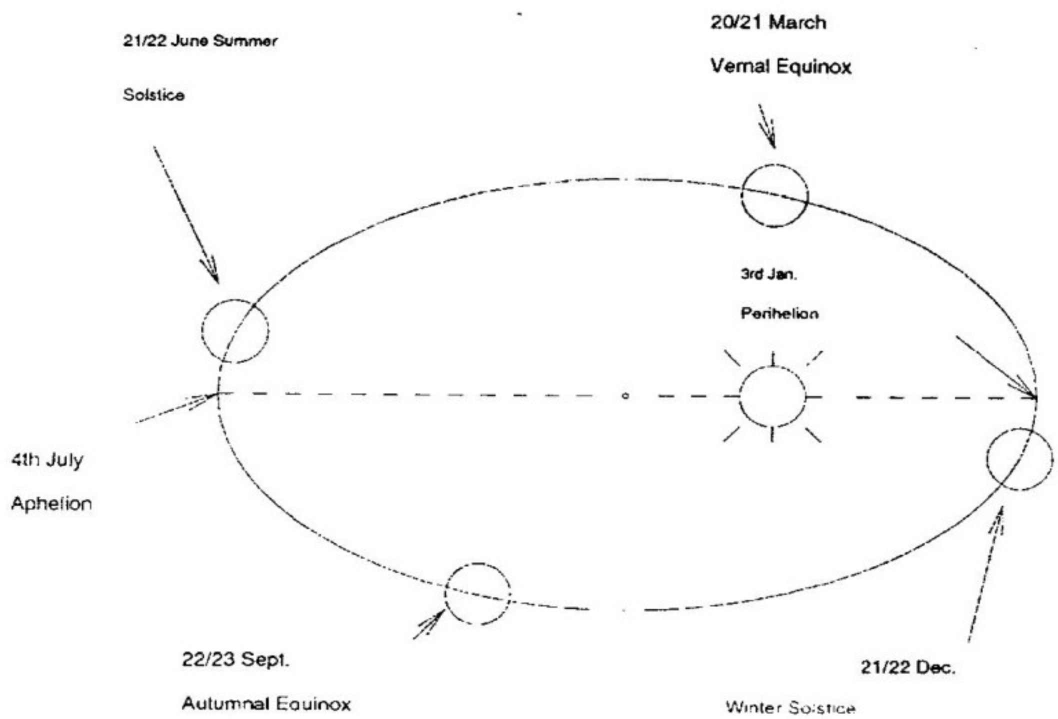

Fig. 1 Motion of the earth around the sun

In order to have a simple mathematical expression giving the earth-sun distance at any given moment, a number of expressions of varying complexities are in use. One such an expression is by Spencer [1] who gives it as the reciprocal of the square of the distance of the earth from the sun, also called the eccentricity correction factor of the earth's orbit, as

$$(d_0/d)^2 = r$$

$$= 1.000110 + 0.034221\text{Cos}\Gamma + 0.001280\text{Sin}\Gamma$$

$$+ 0.000719\text{Cos}2\Gamma + 0.000077\text{Sin}2\Gamma \quad (1)$$

where

$$\Gamma = 2\pi\,(n-1)/365 \text{ radians.}$$

Duffie and Beckman [2] and Iqbal [3] on the other hand say that for most engineering and technological applications in the field of solar energy, a simpler expression

$$(d_0/d)^2 = r$$

$$= 1 + 0.033\text{Cos}(2\pi n/365) \quad (2)$$

is quite adequate.

Because Equation (1) was obtained by performing a Fourier analysis on the measured data, its derivation is impossible without data. Equation (2) also appears to have been obtained by fitting and thus it is also difficult to derive without data. In this paper a novel approach is proposed and uses plane geometrical considerations, Ledermann [4], on the earth's orbit around the sun on the assumption that the earth follows an elliptical path, Iqbal [3] in accordance with Kepler's laws.

A NOVEL EXPRESSION FOR THE EARTH-SUN DISTANCE

In cartesian coordinates, the earth's orbit around the sun, *S*, can be represented by an ellipse whose major axis is of length . X'X, the minor axis of length Y'Ywith an eccenticity of *e* as shown in Figure 2.

Since it is known that the sun is at one of the foci of the elliptical path and that this focus is at a distance of X'S which is equal to $1.017d_0$ when the earth is furthest from the sun and *SX* which is equal to $0.983d_0$ when nearest to the sun, Iqbal [3], then the major axis which is of length X'X becomes $2d_0$.

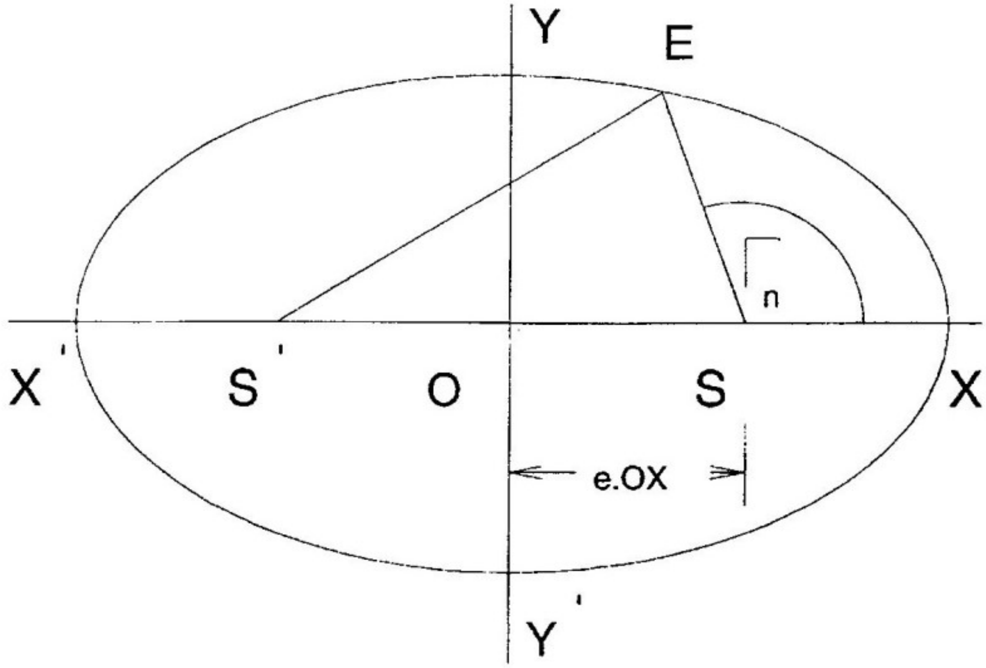

Fig. 2 The properties of an ellipse used to determine the Earth-Sun distance expression.

Because $OS = e(OX)$ and $OS = OX - SX$, then $OS = 0.017d_o$ which gives an eccentricity of $e = 0.017$.

Using the properties of an ellipse where it is known that

$$((1/2)(X'X))^2e^2 = ((1/2)(X'X))^2) - (1/2)(YY'))^2$$

and S'E+ES=X'X , (see Figure 2)

then

$$((1/2)(Y'Y))^2 = ((1/2)(X'X))^2(1 - e^2)$$

giving

$$((1/2)(Y'Y))^2 = d_o^2(1 - e^2)$$

and from triangle S'ES in Figure 2

$$(S'E)^2 = (SE)^2 + (S'S)^2 + 2(SE)(S'S)\mathrm{Cos}\Gamma_n,$$

after applying the cosine rule to the triangle.

But since S'E + ES = X'X and X'X = $2d_o$, S'S = $2d_oe$ and *ES* = d then

$$(S'E)^2 = (2d_o - d)^2$$

giving

$$(2d_o - d)^2 = d^2 + (2d_oe)^2 + 2d(2d_oe)\mathrm{Cos}\Gamma_n$$

which simplifies to

$$d_o/d = (1 + e\mathrm{Cos}\Gamma_n)/(1 - e^2)$$

which gives the ratio $(d_o/d)^2$ as

$$(d_o/d)^2 = (1+2e\mathrm{Cos}\Gamma_n+e^2\mathrm{Cos}^2\Gamma_n)/(1-2e^2+e^4). \quad (3)$$

where the angle *ESX* (Γ_n) varies from day to day and the 1st of January is taken as day one, i.e., n = 1.

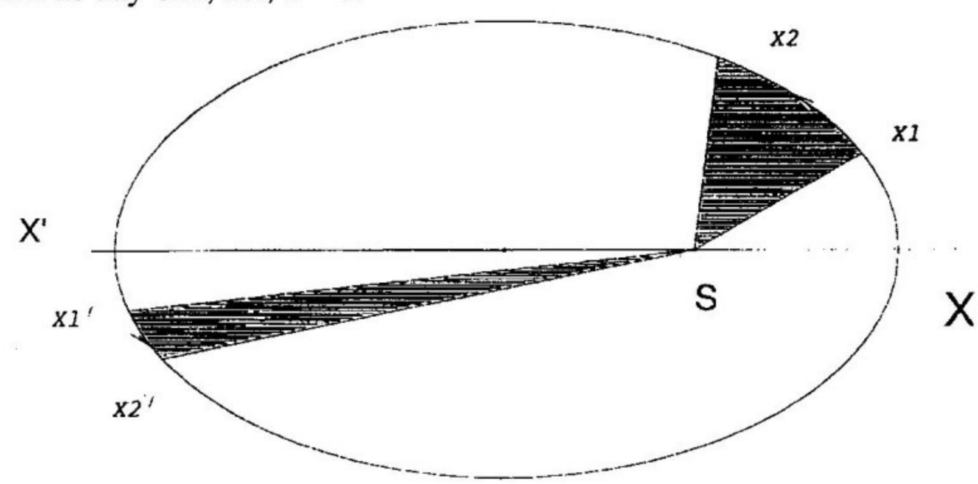

Fig. 3 Keppler's second law in a geometrical perspective

But according to Kepler's Second Law which says that the areal velocity of a planet revolving around the sun is constant (meaning the area swept by the radius vector of a planet per unit time is constant) (see Figure 3), the angle Γ_n, therefore, does not increase in direct proportionality with time. The implication of this law is that the angle Γ_n in addition to varying with the time t, it also depends on the eccentricity e. And as a concequence of Kepler's Second Law therefore, it follows that the speed of the earth around the sun is highest at the position closest to the sun, called the Perihelion, denoted by X , and slowest when furthest from the sun, called the Aphelion, denoted by X', see Figure 4.

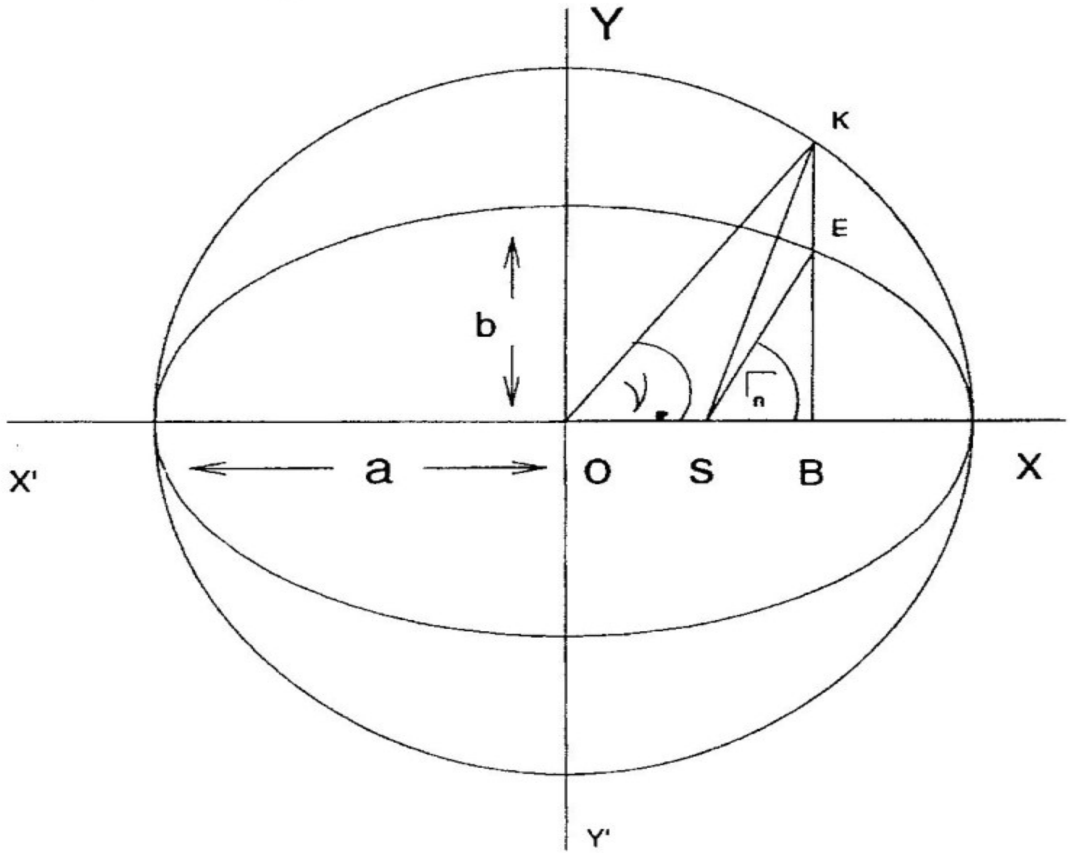

Fig. 4. The properties of an ellipse and a circle used to determine the Earth-Sun distance expression

If we denote the period of revolution by P, the time ellapsed since passing point X by t, the area swept by the radius vector in time t by A', and the total area of the ellipse by A, then according to Kepler's Second Law

$$A'/A = t/P \tag{4}$$

But for an ellipse (see Figure 4) $A = \pi ab$ and equation (4) therefore becomes

$$A = \pi ab/P = A'/t \qquad (5)$$

If we define an angle π to be the angle subtended at the centre of a circle of radius a (where a is the semi-major axis of the earth's orbit) by an arc *XK*, where *K* is a point on the circle obtained by producing a line through *E* which is perpendicular to the line *OSBX* at point *B*, see Figure 4, then

$$(EB)/(KB) = b/a$$

But from Equation (5) the area A' of sector *XSE* from the ellipse is given by

$$A' = \pi\, a\, b\, t/P \qquad (6)$$

while by considering sectors *XSE* and *XSK* the area of *XSE* is also given by

$$A' = (EB/KB)(\text{area of sector XSK})$$

But now if sector *XSK* is given by the difference of areas of sector *XOK* and triangle *SOK* where area of sector is given by

$$XSK = 0.5\upsilon a^2 - (1/2)a^2 e \mathrm{Sin}\, \upsilon$$

then

$$A' = (1/2)ab(\upsilon - e\mathrm{Sin}\, \upsilon). \qquad (7)$$

Equating (6) and (7) we obtain

$$\pi a\, b\, t/P = (1/2)ab(\upsilon - e\mathrm{Sin}\, \upsilon)$$

which gives

$$\upsilon - e\mathrm{Sin}\, \upsilon = 2\pi t/P \qquad (8)$$

Equation (8) above shows that if the eccentricity e were zero, the angle would increase linearly with time as given by the expression on the right hand side. Therefore if we define 2 π t /P to be the angle *M* (analogous to Γ_n except that *M* increases linearly with time from 0 to 2π radians in one period *P*) then equation (8) changes to

$$\upsilon - e\mathrm{Sin}\, \upsilon = M. \qquad (9)$$

which is transcendental in υ and, therefore, for a given *M* the solution for

υ can only be found numerically or graphically.

A look at Figure 4 shows that if d and Γ_n are the heliocentric orbital coordinates of the earth at point *E*, where d is the distance of the earth from the sun and Γ_n is the angle made by vector **d** from the time when **d** was the shortest, then with a semimajor axis of a and an eccentricity of e we have $OB=OS+SE\,\mathrm{Cos}\Gamma_n$ or

$$a\mathrm{Cos}\,\upsilon = ae + d\mathrm{Cos}\Gamma_n. \qquad (10)$$

But because the equation of an ellipse is also given by

$$d = a(1-e^2)/(1+e\mathrm{Cos}\,\Gamma_n) \qquad (11)$$

then we get

$$\mathrm{Cos}\,\upsilon = (e + \mathrm{Cos}\,\Gamma_n)/(1+e\mathrm{Cos}\,\Gamma_n) \qquad (12)$$

which with a few trigonometric manipulations we obtain

$$1 - \mathrm{Cos}\,\upsilon = (1+e)(1+\mathrm{Cos}\,\Gamma_n)/(1+e\mathrm{Cos}\,\Gamma_n)$$

and

$$1 + \mathrm{Cos}\,\upsilon = (1-e)(1-\mathrm{Cos}\,\Gamma_n)/(1+e\mathrm{Cos}\,\Gamma_n)$$

which can be combined to yield

$$\tan^2(\upsilon/2) = (1-e)\tan^2(\Gamma_n/2)/(1+e) \qquad (13)$$

while if Equations (10) and (11) are solved for d in terms of a, e, and υ yields

$$d = a(1 - e\mathrm{Cos}\,\upsilon). \qquad (14)$$

If we now start with a given value of *M*, by using Equation (9) we can find υ which when substituted in Equations (13) and (14) we can obtain Γ_n and d, the orbital coordinates of the earth. The value of Γ_n can then be substituted in equation (3) to obtain the ratio $(d_o/d)^2$.

Comparison of $(d_o/d)^2$ obtained by the three different methods

Because the earth-sun distance can be given in terms of the ratio $(d_o/d)^2$, it is always enough to give this ratio when the earth-sun distance is sought. However whereas one would expect that the currently used values of $(d_o/d)^2$ should not differ from each other, the contrary is observed. Figure 5 shows clearly this discrepancy where it is shown that the calculated values by the two currently accepted methods on daily basis have differences.

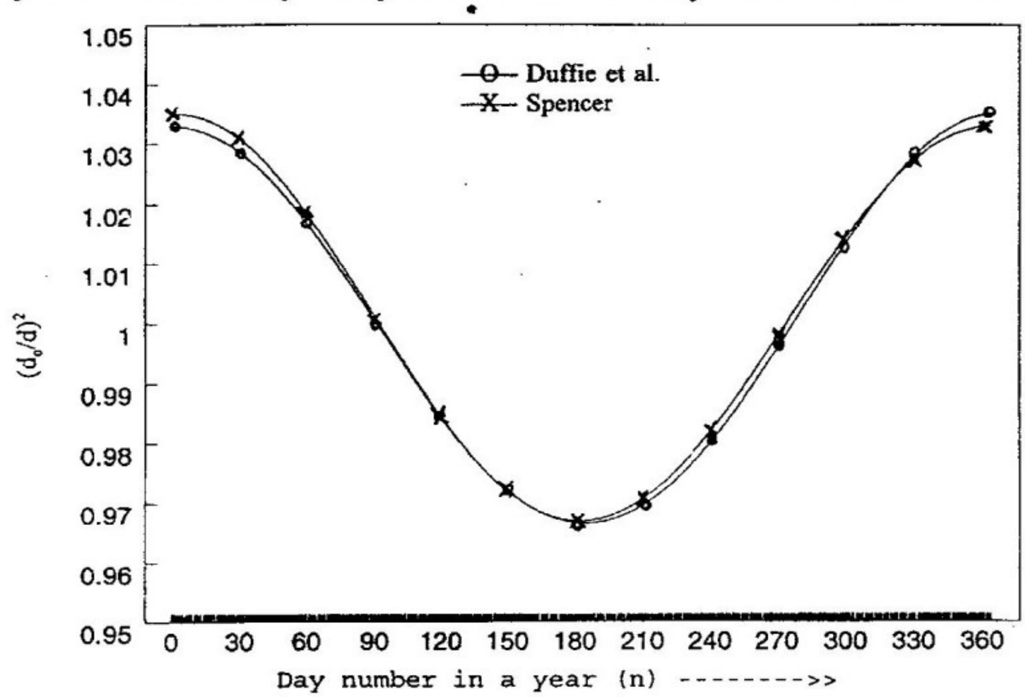

Fig. 5. The ratio $(d_o/d)^2$ by currently used methods.

To show the deviations of the results obtained by the proposed novel method, the results from the two currently accepted methods are plotted together with the newly obtained results as shown in Figure 6. To highlight the deviations, in Figure 7 the differences in the results obtained by the two currently used methods from the results obtained by the proposed novel method are shown.

In Figures 8 and 9 graphs similar to those given in Figures 6 and 7 but this time giving also measured values of $(d_o/d)^2$ at weekly intervals are given. In Figure 8 it is seen that the values obtained by the novel method compare better with the measured values compared to the results by the other methods. Figure 9 furthermore, clearly shows the above observation by giving the deviations from the measured values where it is seen that the novel method has lower deviations than the other two methods.

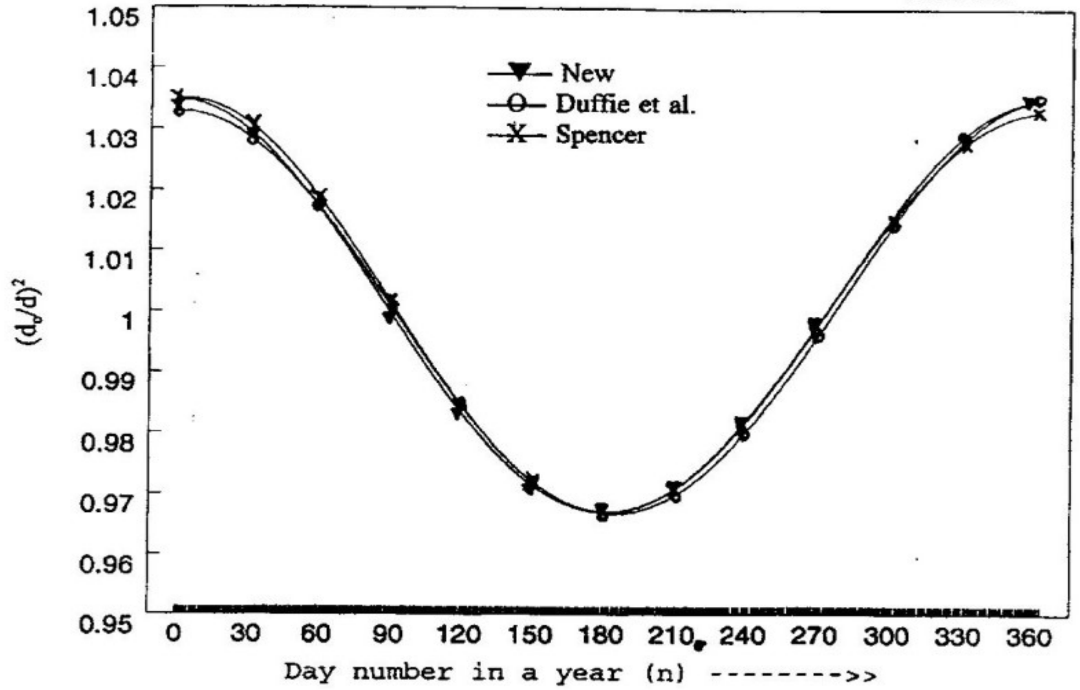

Fig. 6. The ratio $(d_o/d)^2$ by the new and currently used methods

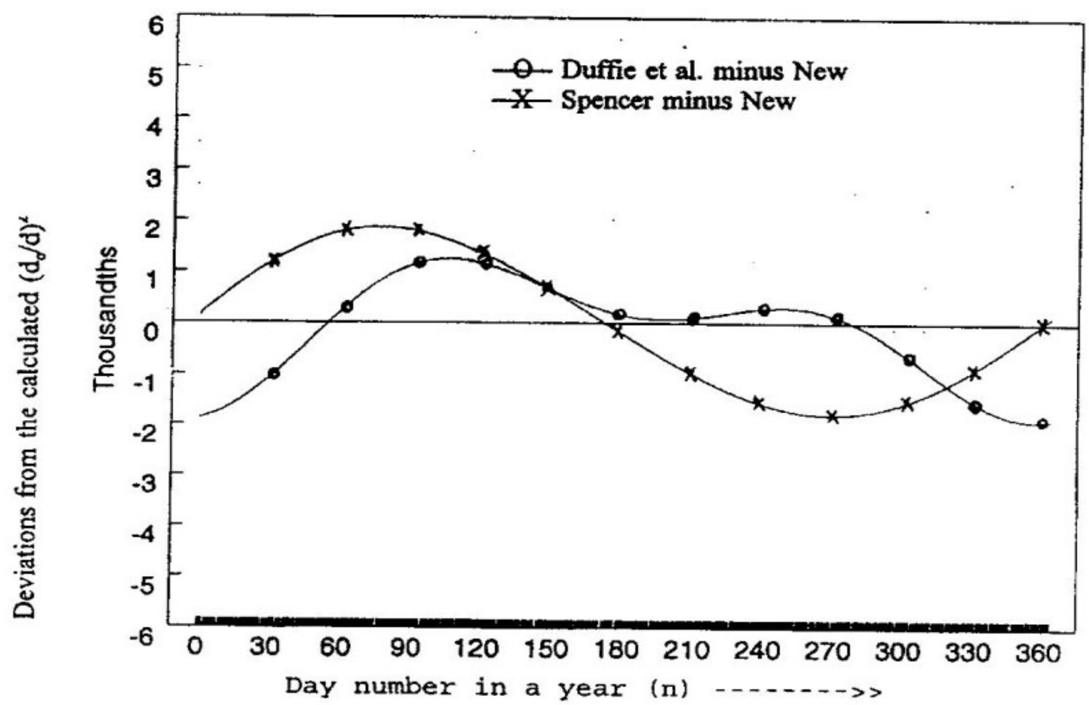

Fig. 7 Deviation of $(d_o/d)^2$ of the currently used methods from the new method

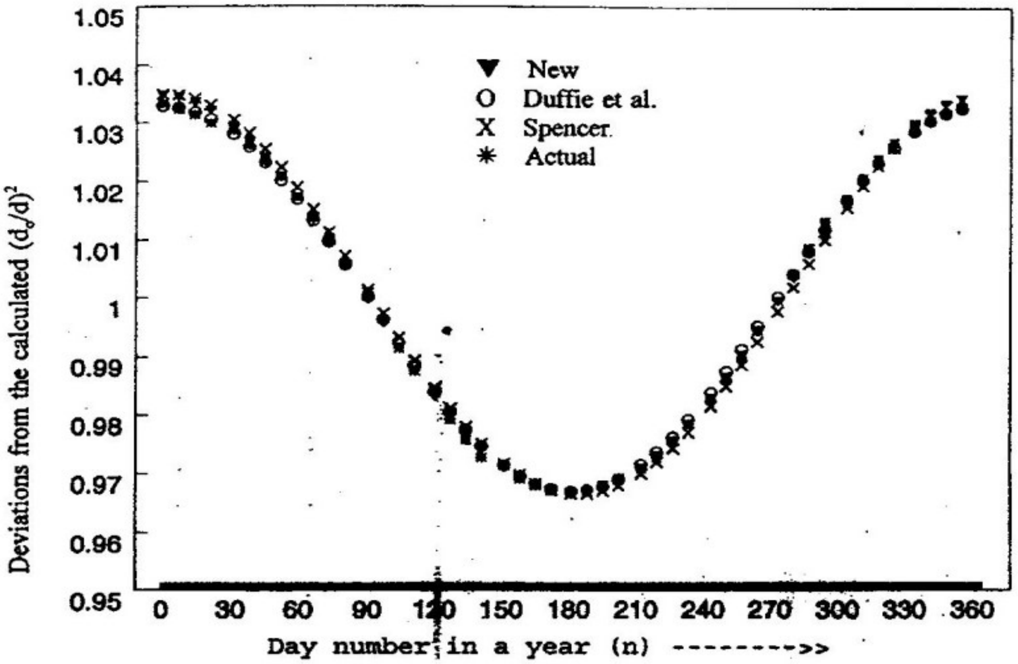

Fig. 8 The ratio $(d_o/d)^2$ by the new and currently used methods at weekly intervals.

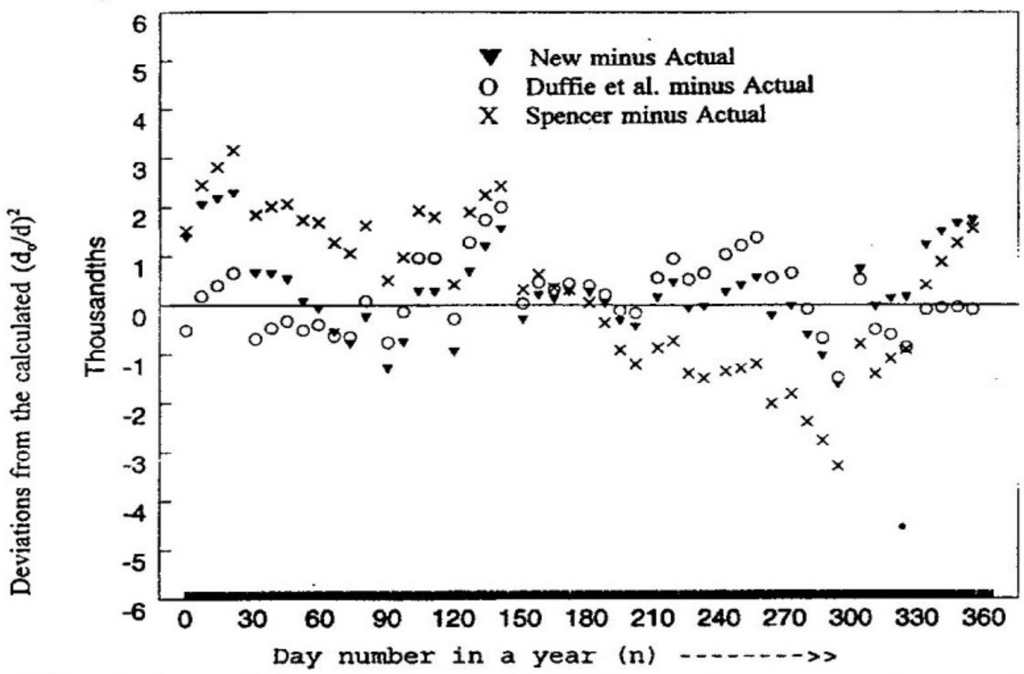

Fig 9 Deviations of the calculated $(d_o/d)^2$ from the measured at weekly intervals

Discusion

The results as shown in Figures 5,6 and 7 clearly show that the currently used methods of calculating $(d_o/d)^2$ as quoted by Spencer [1] and Duffie et al. [2] and Iqbal [3] give results which do not agree with each other. The reason for their differences might be attributed to the way the expressions were derived and the assumptions used. Since the assumptions used in deriving them are not known, their improvement is difficult. A novel way of getting the same ratio therefore is that which is proposed in this paper where it is seen that the only assumption used is that the Kepler 's laws hold and the fact that the period of revolution is 365.25 days.

Using the results obtained by the novel method shows that the deviations of the ratio $(d_o/d)^2$ from the actual values are less than those obtained by the currently used methods. In Figure 8 values of $(d_o/d)^2$ obtained by the three methods and those by measurement at weekly intervals are displayed while in Figure 9 the corresponding deviationsof the values from the measured values are given.

CONCLUSION

The paper has derived a novel expression for the instantaneous earth-sun distance using only Kepler 's laws and some geometrical relations. Unlike in the previous expressions, this paper does not require the user to make any other additional assumptions other than those inherent with the Kepler's laws and therefore they give the user the freedom of determining υ to any accuracy wanted and hence that of $(d_o/d)^2$ by using Equation, (9) only.

LIST OF SYMBOLS

d	instantaneous earth-sun distance
d_o	mean sun-earth distance
e	eccentricity of an ellipse
r	ratio of d_o to d
n	day number in a year, 1st January being taken as n=1
Γ	day angle, angle subtended by the earth-sun distance vector on the sun from some reference earth-sun distance

vector which increases linearly with time

Γ_n day angle, angle subtended by the earth-sun distance vector on the sun from some reference earth-sun distance vector which increases nonlinearly with time (defined in this paper)

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