Uhandisi Journal Vol. 21 No. 2, September 1997, pp 100-106

SENSITIVITY ANALYSIS USING THE SMITH CHART

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ABSTRACT

Smith Chart-based methods for computing the sensitivity of a network or system and for obtaining polar plots of sensitivity as a function of frequency are developed. Examples demonstrate that the use of the Smith Chart simplifies the computation of network and system sensitivities.

INTRODUCTION

Sensitivity is a measure of the change in some performance characteristic of a network or system resulting from a change in the nominal value of one or more of the elements or components of a network or system [1].

There are two definitions of sensitivity: normalized and unnormalized sensitivity. We shall use the symbol S_B^A to represent normalized sensitivity and the symbol $\overline{S}_B^A$ to represent unnormalized sensitivity. Here, the superscript represents the performance characteristic that is changing and the subscript represents the network or system component or element that is causing the change.

The unnormalized sensitivity is the ratio of the change in the performance characteristics, ΔA to the change in the element, ΔB i.e.

$$\overline{S}_B^A = \frac{\Delta A}{\Delta B} \qquad (1)$$

The normalized sensitivity is, on the other hand, defined as

$$S_B^A = \frac{\Delta A/A}{\Delta B/B} \qquad (2)$$

For small changes in A and B, the definitions in (1) and (2) above can be approximated by:

$$\bar{S}_B^A = \frac{dA}{dB} \tag{3}$$

and

$$S_B^A = \frac{B}{A}\frac{dA}{dB} \tag{4}$$

SENSITIVITY ANALYSIS USING THE SMITH CHART

The Smith Chart is a bilinear conformal mapping on the complex ρ-plane of the complex z-plane where ρ and z are related by

$$\rho = \frac{z-1}{z+1}$$

Plotting values of z on the Smith Chart enables one to easily determine the magnitude and phase of ρ. Details on drawing frequency response plots on the Smith Chart are available elsewhere [2].

We shall confine our discussion to performance characteristics which are either filter transfer functions on closed-loop frequency responses of control systems. Extensions to cover other types of sensitivity computations are straight-forward.

To obtain a Smith Chart plot of a filter transfer function, H(s), or that of the open loop or closed-loop frequency response of a control system whose open-loop transfer function is KG(s) we plot on the Smith Chart values of z given by:

$$Z = \left.\frac{1+KG(s)}{1-KG(s)}\right|_{s=j\omega} \tag{5}$$

and

$$z = 2KG(s)+1 \;\Big|_{s=j\omega} \tag{6}$$

respectively

For a transfer function, H(jω), of a filter, we use

$$z = \frac{1+H(j\omega)}{1-H(j\omega)} \tag{7}$$

Smith Chart plots can be used to obtain information about a filter or control system such as stability, magnitude and phase response, etc. [2].

Assuming a Smith Chart plot is available, it can be easily used to compute sensitivity as is shown in Fig.1 below. A parameter is varied and the resulting change in (H(jω) or T(jω) at a given frequency, ω, is determined as shown in Fig.1.

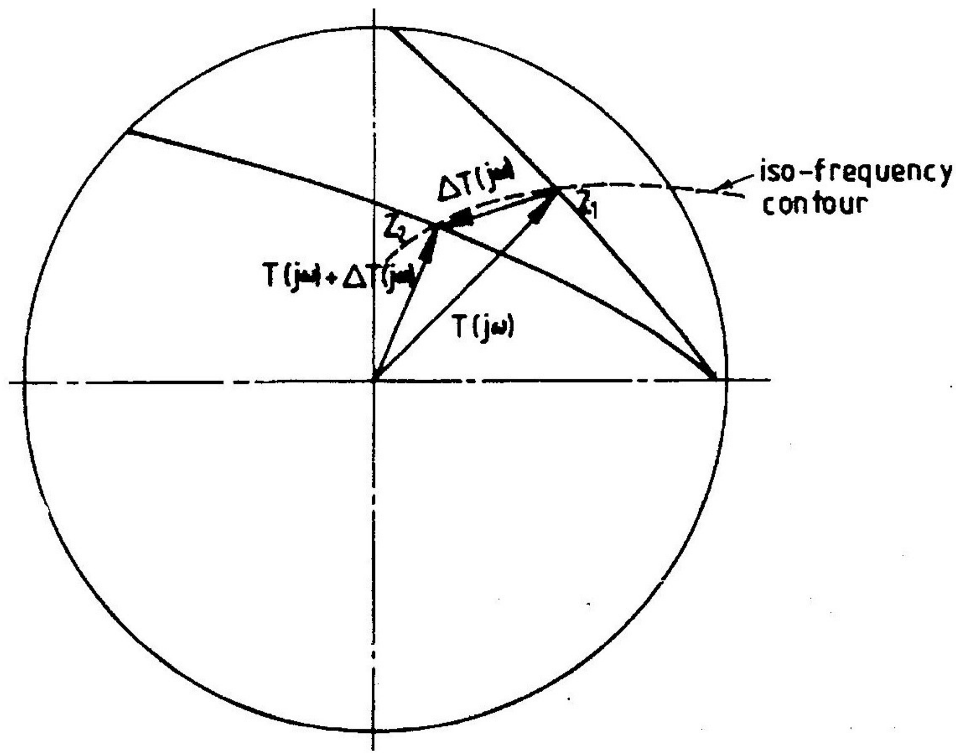

Fig. 1. Determination of Sensitivity on the Smith Chart

Vector addition on the Smith Chart is not as straightforward as it is on the argand diagram. In Fig.1, for example, the point z_1 (the tip of the vector T(jω) lies on the curve obtained when the parameter to be varied is at its nominal value whereas the point z_2 lies on the curve obtained when the parameter has undergone a small change. Points z_1 and z_2, are iso-frequency points and they are joined by the vector T(j ω). In terms of z_1 and z_2 , ΔT(jω) is given by

$$\Delta T(j\omega) = \frac{1}{|z_1|^2}[1+j2(z_1-Re(z_1))] - \frac{1}{|z_2|^2}[1+j2(z_2-Re(z_2))] \quad (8)$$

where $R_c(\bullet)$ represents the real part of its argument. Knowing T(j ω) and the change in the parameter value enables us to use eqns (1) or (2) to compute the sensitivity.

If only the gain K of the open-loop response of a control system, KG(s), is varied then the resulting iso-frequency points z_1 and z_2 are related by

$$z_2 = \frac{K_2}{K_1}(z_1+1)-1 \qquad (9)$$

where K_1 is the original value of gain and K_2 is the new value of gain if both points lie on an inverse plot [2]. Relationships where one point, z_1 or z_2, lies on an inverse plot and the other point lies on a direct plot are given elsewhere.[2].

Example 1. A unity-feedback control system has an open-loop transfer function given by

$$KG(s) = \frac{k}{s(\delta+1)} \qquad (10)$$

The nominal value of K is 10. If K increases by 10 percent, determine the normalized sensitivity of the closed-loop frequency response at a frequency of $\omega = 2$ rads/sec.

The Smith Chart plot for example 1 is shown in Fig. 2.
Using Fig.2 and eqns (8) and (2) we obtain:

$$\Delta T(j\omega) = 0.0971e^{j\,119.1}$$

and

$$S_k^{T(j\omega)} = -0.45 + j0.5$$

This result compared well with the analytical result obtained using eqns (4) and (10) which is:

$$S_k^{T(j\omega)} = -0.5 + j0.5$$

SENSITIVITY CURVES ON THE SMITH CHART

A curve of sensitivity versus frequency can be easily plotted on the Smith Chart. Thus, for example, for a unity-feedback control system whose open-loop transfer function is KG(jω), the plot of the un-normalized sensitivity with respect to K versus frequency is obtained on the Smith Chart by setting.

$$z = \frac{[1+KG(j\omega)]^2 + G(j\omega)}{[1+KG(j\omega)]^2 - G(j\omega)} \qquad (11)$$

whereas that of normalized sensitivity with respect to K versus frequency is obtained by setting.

$$z = \frac{G(j\omega)\,[1+KG(j\omega)]+1}{G(j\omega)\,[1+KG(j\omega)]-1} \tag{12}$$

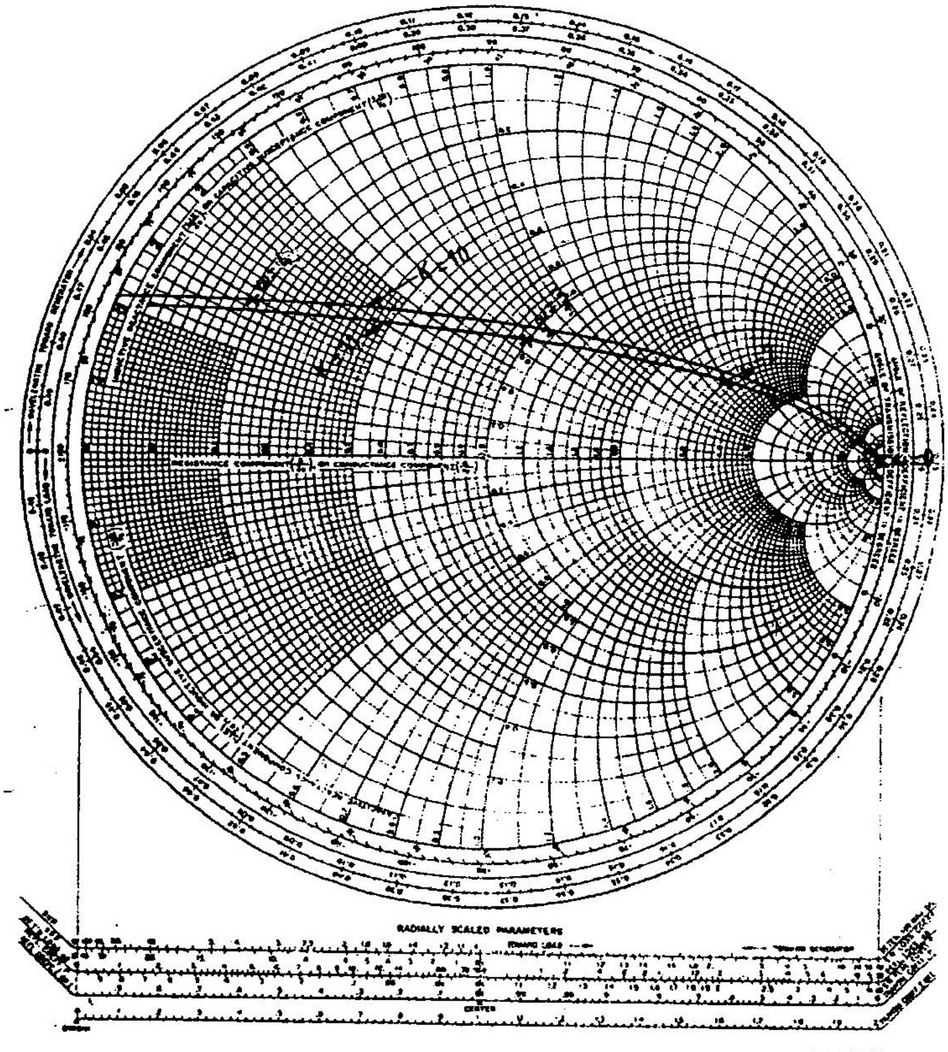

Fig. 2. Smith Chart plot for example 1

It can be easily shown that Smith Chart plots of the sensitivity of an arbitrary transfer function, T(jω), with respect to an arbitrary parameter x, say, can be obtained by setting:

$$z = \frac{1 + \frac{dT(j\omega)}{d\alpha}}{1 - \frac{dT(j\omega)}{d\alpha}} \tag{13}$$

for un-normalized sensitivity and

$$z = \frac{T(j\omega) + \alpha\frac{dT(j\omega)}{dx}}{T(j\omega) - \alpha\frac{dT(j\omega)}{dx}} \tag{14}$$

for normalized sensitivity.

Smith Chart plots of sensitivity versus frequency can be used to easily obtain polar plots or Bode diagrams of sensitivity versus frequency.

SENSITIVITY CURVES FOR SECOND ORDER SYSTEMS

Most Control systems can be adequately approximated by second order systems. In this section we derive a relation for plotting sensitivity versus frequency curves on the Smith Chart for second order systems.

Assume we have a unity-feedback, second order control system whose open-loop transfer function is:

$$KG(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \tag{15}$$

where ζ is the damping coefficient for the system and ω_n is its resonant frequency.

We can easily shown that the normalized sensitivity versus frequency curves on the Smith Chart for the system of equation (15) and for variations with respect to ζ we need to set

$$z = \frac{\left(\frac{\omega}{\omega_n}\right)^2 + j4\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} \tag{16}$$

whereas for variations with respect to ω_n we need to set

$$z = \frac{1 - 3\left(\frac{\omega}{\omega_n}\right)^2 + j4\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{17}$$

CONCLUSION

The use of the Smith Chart to compute and graphically display normalized and un-normalized sensitivity to parameter changes has been demonstrated. Using the differential sensitivity concept we were limited to small changes in a parameter of interest and if that parameter was frequency then we would be limited to a small band of frequencies around the frequency of interest. If one, however, uses the Smith Chart to study the variations of the response of a system or filter to large changes in a parameter, the above limitations are overcome.

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The manuscript was received on 5th September 1996 and accepted for publication on 29th October 1997